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Weakly compact composition operators on vector-valued BMOA

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Abstract

Weak compactness of the analytic composition operator $f \mapsto f \circ \varphi$ is studied on $BMOA(X)$, the space of X-valued analytic functions of bounded mean oscillation, and its subspace $VMOA(X)$, where X is a complex Banach space. It is shown that the composition operator is weakly compact on $BMOA(X)$ if X is reflexive and the corresponding composition operator is compact on the scalar-valued $BMOA$. A concrete example is given which shows that $BMOA(X)$ differs from the weak vector-valued $BMOA$ for infinite dimensional Banach spaces X.

Keywords: Weakly compact composition operators; Bounded mean oscillation; Vector-valued analytic functions

1. Introduction

Let φ be an analytic self-map of the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and X a complex Banach space. The composition operator C_φ induced by φ is the linear map

$$C_\varphi : f \mapsto f \circ \varphi$$

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defined on the linear space of all analytic functions $f:\mathbb{D}\to X$. A fundamental problem concerning composition operators is to relate operator theoretic properties of C_φ to function theoretic properties of φ when restricted to a suitable Banach space of analytic functions. Compactness and weak compactness of C_φ have been studied on many classical Banach spaces such as Hardy spaces (see [13,28]), Bergman and Bloch spaces, and *BMOA* [9,12,29,32]. Recently these studies have been extended by considering weak compactness of composition operators on spaces of X-valued analytic functions, where X is an arbitrary complex Banach space. In [8,25] results of this type have been obtained, e.g., for vector-valued Hardy spaces $H^p(X)$ and vector-valued (weighted) Bergman and Bloch spaces. In this paper we consider composition operators C_φ on $BMOA(X)$, the space of X-valued analytic functions of bounded mean oscillation.

The main goal of this paper is to show that if the map $\varphi:\mathbb{D}\to\mathbb{D}$ induces a compact composition operator on *BMOA* and X is a reflexive complex Banach space, then C_φ is weakly compact on $BMOA(X)$ (see Theorem 7). As a consequence, we obtain a characterization of the weakly compact composition operators C_φ on $BMOA(X)$ under some restrictions on φ for reflexive Banach spaces X. The idea of the main theorem is to generalize the characterization due to Smith [29] of the compact composition operators on *BMOA* to the vector-valued case. For this aim we apply methods developed by Liu, Saksman and Tylli [25].

In the final section we consider a weak version of the vector-valued *BMOA* denoted by $wBMOA(X)$. By a general result due to Bonet, Domański and Lindström [8] the counterpart for $wBMOA(X)$ of our main theorem holds: If C_φ is compact on *BMOA* and X is reflexive, then C_φ is weakly compact on $wBMOA(X)$. We provide a concrete example demonstrating that the spaces $BMOA(X)$ and $wBMOA(X)$ are different for any infinite dimensional Banach space X. Thus our main theorem applies to a different setting compared to [8]. An example of this type was earlier given in [22] in the case where X is an infinite dimensional Hilbert space.

2. Preliminaries on vector-valued *BMOA*

In the sequel X will always be a complex Banach space. Let $H^p(X)$ denote the Hardy space of analytic functions $f:\mathbb{D}\to X$ such that

$$\|f\|_{H^p(X)}^p=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\big\|f\big(re^{i\theta}\big)\big\|_X^p\,d\theta<\infty\quad\text{for } 1\leqslant p<\infty,$$

and $\|f\|_{H^\infty(X)}=\sup_{z\in\mathbb{D}}\|f(z)\|_X<\infty$ for $p=\infty$. One useful way to define the vector-valued *BMOA* space is to view it as the Möbius invariant version of $H^1(X)$ (cf. [2]): An analytic function $f:\mathbb{D}\to X$ belongs to $BMOA(X)$ if and only if

$$\|f\|_{*,X}=\sup_{a\in\mathbb{D}}\big\|f\circ\sigma_a-f(a)\big\|_{H^1(X)}<\infty,$$

where σ_a is the Möbius transformation $\sigma_a(z)=(a-z)/(1-\bar{a}z)$ for $a\in\mathbb{D}$. The norm in $BMOA(X)$ is given by $\|f\|_{BMOA(X)}=\|f(0)\|_X+\|f\|_{*,X}$.

An alternative way to consider the vector-valued *BMOA* is to view it as the space of Poisson extensions of the vector-valued *BMO* functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$ having vanishing negative Fourier coefficients (cf. [5,6]). Let $BMOA_{\mathbb{T}}(X)$ denote the space of such functions equipped with the *BMO* norm on the boundary. By modifying the scalar arguments, one sees that $BMOA_{\mathbb{T}}(X) \subset BMOA(X)$, and that the norms of these spaces are equivalent when restricted to $BMOA_{\mathbb{T}}(X)$. Moreover, $BMOA_{\mathbb{T}}(X)$ can be identified (up to equivalent norms) with the closed subspace of $BMOA(X)$ consisting precisely of the functions $f \in BMOA(X)$ for which the radial limit function $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists almost everywhere on $\mathbb{T}$ (see, e.g., [19, Satz 2.7] for the analogous result for vector-valued Hardy spaces).

For general Banach spaces X the radial limits of $f \in BMOA(X)$ need not exist almost everywhere on $\mathbb{T}$. In fact, the identity $BMOA(X) = BMOA_{\mathbb{T}}(X)$ holds if and only if X has the analytic Radon–Nikodým property (ARNP). Recall that X has the ARNP if and only if the radial limits of every $f \in H^p(X)$ exist almost everywhere on $\mathbb{T}$, and this fact is independent of $p \in [1, \infty]$ [3,10]. The same fact holds also for $BMOA(X)$ because of the inclusions $H^\infty(X) \subset BMOA(X) \subset H^1(X)$.

We define the space $VMOA(X)$ as the closure in $BMOA(X)$ of the X-valued analytic polynomials, that is, the functions of the form $p(z) = \sum_{k=0}^{N} x_k z^k$ where $x_k \in X$. Clearly $VMOA(X) \subset BMOA_{\mathbb{T}}(X)$. In fact, $VMOA(X)$ consists of the extensions of the X-valued *VMO* functions on $\mathbb{T}$ having vanishing negative Fourier coefficients. By modifying the scalar arguments (see, for instance, [18]), we see that $f \in VMOA(X)$ if and only if $f \in BMOA_{\mathbb{T}}(X)$ and

$$\lim_{|a|\to 0} \left\| f \circ \sigma_a - f(a) \right\|_{H^1(X)} = 0.$$

We denote for simplicity $H^p = H^p(\mathbb{C})$, $BMOA = BMOA(\mathbb{C})$, $VMOA = VMOA(\mathbb{C})$, and $\|f\|_* = \|f\|_{*,\mathbb{C}}$ in the scalar case $X = \mathbb{C}$.

Various questions about vector-valued *BMOA* functions have been studied earlier by O. Blasco (see, for instance, [5–7]). The reader is referred to [2,17,18] for the scalar *BMOA* and *VMOA* theory.

3. Boundedness of C_φ on $BMOA(X)$

It is well known that for every analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ the composition operator $C_\varphi : f \mapsto f \circ \varphi$ is bounded on *BMOA*. This fact was first noticed by Stephenson [31, Theorem 3] (see also [1, Theorem 12]). We include here for completeness a proof that C_φ is bounded on $BMOA(X)$ for any complex Banach space X. It is possible to generalize Stephenson's argument to the vector-valued case (this is guaranteed by the boundedness of the composition operator on $H^1(X)$ (see [25, Proposition 1] or [21, Theorem 1])). We give a slightly different argument, in the scalar case due to Smith [29, p. 2716], which motivates our study of weak compactness in the following section. The argument is basically Littlewood's inequality applied to a formula due to Stanton for subharmonic functions.

We first recall some auxiliary concepts. Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic and $0<r\leqslant 1$. The partial Nevanlinna counting function $N_r(\varphi,\cdot):\mathbb{D}\to\mathbb{R}$ is defined by

$$N_r(\varphi,z)=\sum_{w\in\varphi^{-1}(z)}\log^+\left(\frac{r}{|w|}\right)$$

for $z\in\mathbb{D}\setminus\{\varphi(0)\}$, each point in the preimage $\varphi^{-1}(z)$ of $z\in\mathbb{D}$ being repeated according to its multiplicity. Moreover, we put $N_r(\varphi,\varphi(0))=0$. The standard Nevanlinna counting function is given by $N(\varphi,z)=N_1(\varphi,z)=\sum_{w\in\varphi^{-1}(z)}\log(1/|w|)$. We refer to, e.g., [28, Chapter 10] for the properties of the (partial) Nevanlinna counting function. For any complex Banach space X and analytic function $f:\mathbb{D}\to X$, the function $z\mapsto\|f(z)\|_X$ is subharmonic on $\mathbb{D}$. Thus we may define the distributional Laplacian $\Delta\|f\|_X$ of $\|f\|_X$, which is a positive measure on $\mathbb{D}$, by setting

$$\int_{\mathbb{D}}\psi(w)d\big(\Delta\|f\|_X\big)(w)=\frac{1}{2\pi}\int_{\mathbb{D}}\big\|f(w)\big\|_X\Delta\psi(w)\,dA(w)$$

for every test function $\psi\in C_0^\infty(\mathbb{D})$, where dA denotes the Lebesgue area measure on $\mathbb{D}$. The following lemma states a special case of Stanton's formula [30, Theorem 2], and it will be needed several times in the sequel.

Lemma 1 [25, pp. 300–301]. *Let $f:\mathbb{D}\to X$ and $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic functions,* $0<r<1$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\big\|(f\circ\varphi)\big(re^{i\theta}\big)\big\|_X\,d\theta=\big\|f\big(\varphi(0)\big)\big\|_X+\int_{\mathbb{D}}N_r(\varphi,w)\,d\big(\Delta\|f\|_X\big)(w),$$

$$\|f\circ\varphi\|_{H^1(X)}=\big\|f\big(\varphi(0)\big)\big\|_X+\int_{\mathbb{D}}N(\varphi,w)\,d\big(\Delta\|f\|_X\big)(w).$$

The special case $\varphi(z)\equiv z$ yields the identities

$$\frac{1}{2\pi}\int_0^{2\pi}\big\|f\big(re^{i\theta}\big)\big\|_X\,d\theta=\big\|f(0)\big\|_X+\int_{\mathbb{D}}\log^+\left(\frac{r}{|w|}\right)d\big(\Delta\|f\|_X\big)(w),$$

$$\|f\|_{H^1(X)}=\big\|f(0)\big\|_X+\int_{\mathbb{D}}\log\left(\frac{1}{|w|}\right)d\big(\Delta\|f\|_X\big)(w).$$

The following estimates are not difficult to obtain by using the Cauchy integral formula (see, for instance, [18, p. 95]).

Lemma 2. *Let $f:\mathbb{D}\to X$ be analytic and $z\in\mathbb{D}$. Then*

$$\big\|f(z)-f(0)\big\|_X\leqslant\min\left\{\frac{|z|}{1-|z|}\|f\|_{H^1(X)},\ \frac{1}{2}\log\frac{1+|z|}{1-|z|}\|f\|_{*,X}\right\}.$$

We are now ready to prove that every composition operator C_φ is bounded on $BMOA(X)$ for any complex Banach space X.

Proposition 3. *Let φ be an analytic self-map of the unit disk. Then $\|f\circ\varphi\|_{*,X}\leqslant\|f\|_{*,X}$ and $C_\varphi : BMOA(X)\to BMOA(X)$ is bounded with*

$$\|C_\varphi\|\leqslant 1+\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

Proof. For any function $f\in H^1(X)$ and $a\in\mathbb{D}$ one has

$$\big\|f\circ\varphi\circ\sigma_a-f\big(\varphi(a)\big)\big\|_{H^1(X)}=\int_{\mathbb{D}}N(\varphi\circ\sigma_a,w)\,d\big(\Delta\big\|f-f\big(\varphi(a)\big)\big\|_X\big)(w),$$

by Lemma 1. By Littlewood's inequality [13, Theorem 2.29], it holds that $N(\varphi\circ\sigma_a,w)\leqslant\log(1/|\sigma_{\varphi(a)}(w)|)=N(\sigma_{\varphi(a)},w)$ for $w\in\mathbb{D}\setminus\{\varphi(a)\}$. Hence, by applying Lemma 1 once more, one obtains

$$\begin{aligned}\big\|f\circ\varphi\circ\sigma_a-f\big(\varphi(a)\big)\big\|_{H^1(X)}&\leqslant\int_{\mathbb{D}}N(\sigma_{\varphi(a)},w)\,d\big(\Delta\big\|f-f\big(\varphi(a)\big)\big\|_X\big)(w)\\&=\big\|f\circ\sigma_{\varphi(a)}-f\big(\varphi(a)\big)\big\|_{H^1(X)}\\&\leqslant\sup_{b\in\mathbb{D}}\big\|f\circ\sigma_b-f(b)\big\|_{H^1(X)},\end{aligned}$$

so that the inequality $\|f\circ\varphi\|_{*,X}\leqslant\|f\|_{*,X}$ holds for $f\in BMOA(X)$. Thus

$$\begin{aligned}\|C_\varphi f\|_{BMOA(X)}&=\|f\circ\varphi\|_{*,X}+\big\|f\big(\varphi(0)\big)\big\|_X\\&\leqslant\|f\|_{*,X}+\big\|f(0)\big\|_X+\big\|f\big(\varphi(0)\big)-f(0)\big\|_X\\&\leqslant\left(1+\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{BMOA(X)},\end{aligned}$$

by Lemma 2. $\square$

Remark 4. The composition operator C_φ maps the space $BMOA_{\mathbb{T}}(X)$ into itself for any Banach space X. To see this, it is enough to verify that the radial boundary function $(f\circ\varphi)^*$ exists almost everywhere on $\mathbb{T}$ whenever $f\in H^1_{\mathbb{T}}(X)$, where $H^1_{\mathbb{T}}(X)$ is the subspace of $H^1(X)$ consisting of the functions for which the radial limit function exists almost everywhere on $\mathbb{T}$. But this follows from the known facts that $p\circ\varphi\in H^1_{\mathbb{T}}(X)$ for every analytic X-valued polynomial p, and these polynomials form a dense subset of $H^1_{\mathbb{T}}(X)$ (see, for instance, [19, p. 57]).

It is well known that $C_\varphi(VMOA)\subset VMOA$ if and only if $\varphi\in VMOA$ [1, Theorem 12]. We include the vector-valued argument for completeness.

Corollary 5. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic self-map of the unit disk. Then $C_\varphi(VMOA(X))\subset VMOA(X)$ if and only if $\varphi\in VMOA$.*

Proof. Suppose that C_φ maps $VMOA(X)$ into itself. In particular, then $C_\varphi(x_0 z) = x_o\varphi \in VMOA(X)$, where $x_0 \in X$ is non-zero. Clearly this implies that $\varphi \in VMOA$. Conversely, suppose that $\varphi \in VMOA$. Then

$$\lim_{|a|\to 0} \left\| p \circ \varphi \circ \sigma_a - p\big(\varphi(a)\big) \right\|_{H^1(X)} = 0$$

for every analytic X-valued polynomial p (by the proof of [1, Theorem 12]). By Fatou's theorem, $p \circ \varphi \in BMOA_{\mathbb{T}}(X)$, so that $p \circ \varphi \in VMOA(X)$ for every analytic X-valued polynomial p. Since such polynomials are dense in $VMOA(X)$, it follows that C_φ maps $VMOA(X)$ into itself. $\square$

4. Weak compactness of C_φ on $BMOA(X)$

Recall that a bounded linear map $T : X \to X$ is called compact (respectively weakly compact) if it maps the closed unit ball of X onto a relatively compact (respectively relatively weakly compact) set in X. It was noted in [25, p. 296] that C_φ can be compact on $H^p(X)$ only if X is finite dimensional and C_φ is compact on H^p (here $1 \leqslant p \leqslant \infty$). Moreover, if the composition operator is weakly compact on $H^p(X)$, then X must be reflexive. These facts actually hold for various spaces of vector-valued analytic functions [8, Proposition 1] including $BMOA(X)$.

Fact 6. *Suppose that $\mathcal{J}$ is an operator ideal such that the composition operator $C_\varphi : BMOA(X) \to BMOA(X)$ belongs to $\mathcal{J}$. Then the identity operator* $\mathrm{id} : X \to X$ *and the composition operator $C_\varphi : BMOA \to BMOA$ belong to $\mathcal{J}$.*

We refer to [27] for the definition of an operator ideal. Consequently, if C_φ is weakly compact on $BMOA(X)$, then X is reflexive and C_φ is weakly compact on $BMOA$. Our main theorem provides a sufficient condition for the weak compactness of C_φ on $BMOA(X)$.

Theorem 7. *Let X be a reflexive Banach space and suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic map such that $C_\varphi : BMOA \to BMOA$ is compact. Then $C_\varphi : BMOA(X) \to BMOA(X)$ is weakly compact.*

We split the proof of Theorem 7 into two parts. The main idea is to approximate C_φ in the operator norm by suitable weakly compact operators that are provided by Lemma 8 below. For the approximation we need Smith's characterization of the compact composition operators C_φ on $BMOA$. The key step is contained in Proposition 11.

Lemma 8. *There are linear operators $(V_n)_{n=0}^\infty$ on $BMOA(X)$ satisfying the following properties*:

(1) $\|V_n\| \leqslant 3$ *for* $n \geqslant 0$.
(2) *For every* $0 < r < 1$ *one has*

$$\sup_{\|f\|_{BMOA(X)} \leqslant 1} \sup_{|z| \leqslant r} \left\| (f - V_n f)(z) \right\|_X \to 0,$$

as $n \to \infty$.
(3) *If X is reflexive, then V_n is weakly compact on $BMOA(X)$ for $n \geqslant 0$.*

Proof. We use the de la Vallée–Poussin operators V_n defined by setting

$$V_n f(z) = \sum_{k=0}^{n} \hat{f}_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \hat{f}_k z^k$$

for analytic functions $f:\mathbb{D} \to X$ with the Taylor expansion $f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$ (as in [25, Proposition 2]). Note that $V_n f = 2k_{2n-1}(f) - k_{n-1}(f)$, where

$$k_n(f)(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \hat{f}_k z^k = \frac{1}{2\pi} \int_0^{2\pi} K_n(\theta) f\left(ze^{-i\theta}\right) d\theta$$

and K_n is the Fejér kernel (cf. [23, I.2.13]).

The fact that the operators V_n satisfy (1) and (2) is seen as in [25]. We will only check here that (3) holds for every V_n. In fact, by the triangle inequality and the fact $(V_n f)(0) = f(0)$, it is enough to show that $\|k_n(f)\|_{*,X} \leqslant \|f\|_{*,X}$ for $n \geqslant 0$. Let $n \geqslant 0$. Then

$$\begin{aligned}
&\int_0^{2\pi} \left\|k_n(f)\left(\sigma_a\left(re^{it}\right)\right) - k_n(f)(a)\right\|_X \frac{dt}{2\pi} \\
&\quad = \int_0^{2\pi} \left\| \int_0^{2\pi} K_n(\theta)\left[f\left(e^{-i\theta}\sigma_a\left(re^{it}\right)\right) - f\left(e^{-i\theta}a\right)\right] \frac{d\theta}{2\pi} \right\|_X \frac{dt}{2\pi} \\
&\quad \leqslant \int_0^{2\pi} K_n(\theta) \int_0^{2\pi} \left\|f\left(e^{-i\theta}\sigma_a\left(re^{it}\right)\right) - f\left(e^{-i\theta}a\right)\right\|_X \frac{dt}{2\pi} \frac{d\theta}{2\pi} \\
&\quad \leqslant \|f\|_{*,X},
\end{aligned}$$

since $\int_0^{2\pi} K_n(\theta) \frac{d\theta}{2\pi} = 1$ and

$$\int_0^{2\pi} \left\|f\left(e^{-i\theta}\sigma_a\left(re^{it}\right)\right) - f\left(e^{-i\theta}a\right)\right\|_X \frac{dt}{2\pi} \leqslant \sup_{\theta \in [0,2\pi)} \left\|f\left(e^{-i\theta}\,\cdot\,\right)\right\|_{*,X} = \|f\|_{*,X}$$

by the rotation invariance of the seminorm $\|\cdot\|_{*,X}$. We obtain $\|k_n(f)\|_{*,X} \leqslant \|f\|_{*,X}$ by taking the supremum over $r \in (0,1)$ and $a \in \mathbb{D}$. $\square$

Remark 9. In the scalar case the uniform boundedness of the operators k_n on *BMOA* was shown in [20, Theorem 4].

The compact composition operators C_φ on *BMOA* were characterized by Smith [29, Theorem 1.1] as follows. The analytic map $\varphi:\mathbb{D} \to \mathbb{D}$ induces a compact composition operator on *BMOA* if and only if φ satisfies both of the following conditions:

$$\lim_{r\to 1} \sup_{\{a:\ |\varphi(a)|>r\}} \sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0, \tag{1}$$

and

$$\lim_{t\to 1} \sup_{\{a:\ |\varphi(a)|\leqslant R\}} m\bigl(\bigl\{\zeta \in \mathbb{T}:\ \bigl|(\varphi\circ\sigma_a)^*(\zeta)\bigr| > t\bigr\}\bigr) = 0 \tag{2}$$

for every $R < 1$, where m denotes the Lebesgue measure on $\mathbb{T}$. Condition (2) can actually be replaced by the condition

$$\lim_{t\to 1} \sup_{\{a:\ |\varphi(a)|\leqslant R\}} \sup_{0<r<1} m\bigl(\bigl\{\zeta \in \mathbb{T}:\ \bigl|(\varphi\circ\sigma_a)(r\zeta)\bigr| > t\bigr\}\bigr) = 0 \tag{3}$$

for every $R < 1$; that is, C_φ is compact on *BMOA* if and only if both (1) and (3) hold. Since (3) is useful later on, we include for the convenience of the reader a proof of the necessity of (3) (this is a simple modification of the argument in [29, p. 2720]). In fact, if (3) does not hold, then there exist $R < 1$, $\varepsilon > 0$, $t_n < 1$, $r_n \in (0,1)$ and $a_n \in \mathbb{D}$ such that $t_n^n \to 1$, $|\varphi(a_n)| \leqslant R$ and $m(E_n) \geqslant \varepsilon$, where $E_n = \{\zeta:\ |(\varphi\circ\sigma_{a_n})(r_n\zeta)| > t_n\}$. Let $f_n(z) = z^n$, so that $\|f_n\|_{BMOA} \leqslant 1$ and (f_n) converges to 0 uniformly on compact subsets of $\mathbb{D}$. It suffices to check that $C_\varphi f_n$ does not converge to 0 in *BMOA*. Choose n_0 such that $t_n^n \geqslant \frac{2}{3}$ and $R^n \leqslant \frac{1}{3}\varepsilon$ for $n \geqslant n_0$. Then

$$\begin{aligned}\|f_n\circ\varphi\|_{BMOA} &\geqslant \frac{1}{2\pi}\int_{\mathbb{T}} \bigl|(\varphi\circ\sigma_{a_n})^n(r_n\zeta) - \varphi^n(a_n)\bigr|\, dm(\zeta)\\ &\geqslant \frac{1}{2\pi}\int_{E_n} \bigl|(\varphi\circ\sigma_{a_n})(r_n\zeta)\bigr|^n\, dm(\zeta) - R^n\\ &\geqslant t_n^n m(E_n) - \varepsilon/3 \geqslant \varepsilon/3,\end{aligned}$$

for such n, which proves the necessity of (3).

We note that the compact composition operators on *BMOA* were also characterized in [9] in terms of Carleson measures. Compactness of composition operators on *VMOA* was earlier characterized in [32].

The following lemma refines condition (1). It is a slight modification of [29, Lemma 2.1].

Lemma 10. *Let φ be an analytic self-map of the unit disk with $\varphi(0) = 0$. If*

$$\sup_{0<|w|<1} |w|^2 N(\varphi, w) \leqslant \delta^4,$$

where $\delta < e^{-1/2}$, then

$$N(\varphi, z) \leqslant 2\delta^2 \log\bigl(1/|z|\bigr)$$

for $\delta \leqslant |z| < 1$.

Proof. For $\delta \leqslant |z| \leqslant e^{-1/2}$ the estimate $N(\varphi, z) \leqslant \delta^2 \leqslant 2\delta^2 \log(1/|z|)$ follows from the assumption. For $r \in (0,1)$ the subharmonic function $N_r(\varphi, z)$ is bounded by the harmonic function $2e\delta^4 \log(1/|z|)$ on the annulus $\{w \in \mathbb{D}:\ e^{-1/2} < |w| < 1\}$, by the assumption and the fact that $N_r(\varphi, z) \leqslant N(\varphi, z)$. Thus

$$N(\varphi, z) = \lim_{r\to 1} N_r(\varphi, z) \leqslant 2e\delta^4 \log\bigl(1/|z|\bigr) \leqslant 2\delta^2 \log\bigl(1/|z|\bigr)$$

for $e^{-1/2} < |z| < 1$. $\square$

We are now ready to prove the key step of Theorem 7.

Proposition 11. *Let φ be an analytic self-map of the unit disk satisfying conditions* (1) *and* (3). *Then*

$$\|C_\varphi - C_\varphi V_n\| \to 0$$

as $n \to \infty$, where the operators V_n are those of Lemma 8.

Proof. Let $\varepsilon > 0$ and let $f \in BMOA(X)$ be arbitrary. We need to show that there exists $n_0 \in \mathbb{N}$ so that

$$\big\|C_\varphi(I - V_n)f\big\|_{BMOA(X)} \leqslant \varepsilon\|f\|_{BMOA(X)}$$

for every $n \geqslant n_0$, where I is the identity operator on $BMOA(X)$. We introduce the following abbreviations:

- $S_n = I - V_n$,
- $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$,
- $g_{n,a} = (S_n f) \circ \sigma_{\varphi(a)} - (S_n f)(\varphi(a))$,

for $n \geqslant 0$ and $a \in \mathbb{D}$. Note that $\|g_{n,a}\|_{H^1(X)} \leqslant \|S_n f\|_{*,X} \leqslant 4\|f\|_{BMOA(X)}$ for $n \geqslant 0$, by Lemma 8(1). By Lemma 8(2), one has $\|(C_\varphi S_n f)(0)\|_X = \|(S_n f)(\varphi(0))\|_X \leqslant \varepsilon\|f\|_{BMOA(X)}$ for n large enough. Hence, according to the identity $(\sigma_{\varphi(a)} \circ \sigma_{\varphi(a)})(z) = z$, it suffices to show that

$$\sup_{a\in\mathbb{D}} \|g_{n,a} \circ \varphi_a\|_{H^1(X)} = \|C_\varphi S_n f\|_{*,X} \leqslant \varepsilon\|f\|_{BMOA(X)}, \tag{4}$$

for $n \geqslant n_0$. Choose $\delta = \delta(\varepsilon) \in (0, \frac{1}{4})$ such that $\max\{8\delta^2,\ 48\delta\log(1/\delta)\} < \varepsilon$. By the assumption that φ satisfies conditions (1) and (3) there exist a number $R = R(\varepsilon) \in (0,1)$ such that

$$\sup_{0<|w|<1} |w|^2 N(\varphi_a, w) < \delta^4 \tag{5}$$

for every $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$, and a number $t_0 = t_0(\varepsilon) \in (0,1)$ such that

$$m\big(\big\{\zeta \in \mathbb{T}\colon \big|(\varphi \circ \sigma_a)(r\zeta)\big| > t_0\big\}\big) < \varepsilon^2 \tag{6}$$

for every $r \in (0,1)$ and $a \in \mathbb{D}$ satisfying $|\varphi(a)| \leqslant R$.

Consider first $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$. From Lemma 1 and the fact that $g_{n,a}(\varphi_a(0)) = 0$ we get

$$\begin{aligned}\|g_{n,a} \circ \varphi_a\|_{H^1(X)} = &\int\limits_{\delta\leqslant|w|<1} N(\varphi_a, w)\,d\big(\Delta\|g_{n,a}\|_X\big)(w)\\ &+ \int\limits_{|w|<\delta} N(\varphi_a, w)\,d\big(\Delta\|g_{n,a}\|_X\big)(w) =: A + B.\end{aligned}$$

From (5) and Lemma 10 we get the estimate $N(\varphi_a, w) \leqslant 2\delta^2 \log(1/|w|)$ for $\delta \leqslant |w| < 1$. Using Lemma 1 once more, and recalling the choice of δ, we have

$$A \leqslant 2\delta^2 \int\limits_{\delta \leqslant |w| < 1} \log\left(\frac{1}{|w|}\right) d\big(\Delta \|g_{n,a}\|_X\big)(w) \leqslant 2\delta^2 \|g_{n,a}\|_{H^1(X)} \leqslant \varepsilon \|f\|_{BMOA(X)}.$$

To estimate B, note that $2\log(2\delta/|w|) \geqslant 1$ and $\log(1/\delta) \geqslant 1$ for $|w| < \delta < \frac{1}{4}$. From these estimates and Littlewood's inequality [13, Theorem 2.29] we get

$$N(\varphi_a, w) \leqslant \log\left(\frac{1}{|w|}\right) \leqslant \log\left(\frac{2\delta}{|w|}\right) + \log\left(\frac{1}{\delta}\right) \leqslant 3\log\left(\frac{1}{\delta}\right)\log\left(\frac{2\delta}{|w|}\right),$$

for $0 < |w| < \delta$. Thus

$$\begin{aligned} B &\leqslant 3\log(1/\delta) \int\limits_{|w|<\delta} \log\left(\frac{2\delta}{|w|}\right) d\big(\Delta \|g_{n,a}\|_X\big)(w) \\ &\leqslant 3\log(1/\delta) \int\limits_{\mathbb{D}} \log^+\left(\frac{2\delta}{|w|}\right) d\big(\Delta \|g_{n,a}\|_X\big)(w). \end{aligned}$$

From Lemmas 1 and 2 we get that

$$\begin{aligned} B &\leqslant \frac{3\log(1/\delta)}{2\pi} \int\limits_0^{2\pi} \big\| g_{n,a}\big(2\delta e^{i\theta}\big) - g_{n,a}(0)\big\|_X \, d\theta \\ &\leqslant 3\log(1/\delta) \frac{2\delta}{1-2\delta} \|g_{n,a}\|_{H^1(X)} \\ &\leqslant 12\delta \log(1/\delta) \|g_{n,a}\|_{H^1(X)}, \end{aligned}$$

so that $B \leqslant \varepsilon \|f\|_{BMOA(X)}$ in view of the choice of δ. Consequently,

$$\|g_{n,a} \circ \varphi_a\|_{H^1(X)} \leqslant A + B \leqslant 2\varepsilon \|f\|_{BMOA(X)}, \tag{7}$$

for $a \in \mathbb{D}$ satisfying $|\varphi(a)| > R$.

Consider next $a \in \mathbb{D}$ satisfying $|\varphi(a)| \leqslant R$. By Lemma 8(2), there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ so that for every $n \geqslant n_0$ and $|z| \leqslant t_0$ we have

$$\max\big\{ \big\|(S_n f)(z)\big\|_X, \big\|(S_n f)\big(\varphi(a)\big)\big\|_X \big\} \leqslant \varepsilon \|f\|_{BMOA(X)}.$$

Let $r \in (0,1)$ and put $E = \{\zeta \in \mathbb{T}\colon\ |(\varphi \circ \sigma_a)(r\zeta)| > t_0\}$, so that $m(E) < \varepsilon^2$ by (6). Then

$$\begin{aligned} &\frac{1}{2\pi} \int\limits_{\mathbb{D}\setminus E} \big\|(g_{n,a} \circ \varphi_a)(r\zeta)\big\|_X \, dm(\zeta) \\ &\quad = \frac{1}{2\pi} \int\limits_{\mathbb{D}\setminus E} \big\|\big((S_n f) \circ \varphi \circ \sigma_a\big)(r\zeta) - (S_n f)\big(\varphi(a)\big)\big\|_X \, dm(\zeta) \\ &\quad \leqslant \sup_{|z| \leqslant t_0} \big\|(S_n f)(z)\big\|_X + \big\|(S_n f)\big(\varphi(a)\big)\big\|_X \leqslant 2\varepsilon \|f\|_{BMOA(X)}, \end{aligned}$$

for $n \geqslant n_0$. On the other hand,

$$\begin{aligned}
&\frac{1}{2\pi}\int_E \big\|(g_{n,a}\circ\varphi_a)(r\zeta)\big\|_X\,dm(\zeta)\\
&\quad\leqslant m(E)^{1/2}\biggl(\frac{1}{2\pi}\int_{\mathbb{T}} \big\|(g_{n,a}\circ\varphi_a)(r\zeta)\big\|_X^2\,dm(\zeta)\biggr)^{1/2}\\
&\quad\leqslant \varepsilon\big\|(S_nf)\circ\varphi\circ\sigma_a-(S_nf)\big(\varphi(a)\big)\big\|_{H^2(X)}
\end{aligned}$$

by Hölder's inequality and (6). By the analytic John–Nirenberg theorem [2, p. 15], which also holds in the vector-valued setting (with a similar proof as in the scalar case), there exists a constant C such that

$$\begin{aligned}
&\frac{1}{2\pi}\int_E \big\|(g_{n,a}\circ\varphi_a)(r\zeta)\big\|_X\,dm(\zeta)\\
&\quad\leqslant \varepsilon\sup_{b\in\mathbb{D}}\big\|(S_nf)\circ\varphi\circ\sigma_b-(S_nf)\big(\varphi(b)\big)\big\|_{H^2(X)}\\
&\quad\leqslant C\varepsilon\sup_{b\in\mathbb{D}}\big\|(S_nf)\circ\varphi\circ\sigma_b-(S_nf)\big(\varphi(b)\big)\big\|_{H^1(X)}\\
&\quad= C\varepsilon\|S_nf\circ\varphi\|_{*,X}\leqslant C\varepsilon\|S_nf\|_{*,X}\leqslant 4C\varepsilon\|f\|_{BMOA(X)},
\end{aligned}$$

where the last inequalities followed from Proposition 3 and Lemma 8(1). By combining these estimates and taking the supremum over $r\in(0,1)$, we obtain

$$\|g_{n,a}\circ\varphi_a\|_{H^1(X)}\leqslant(2+4C)\varepsilon\|f\|_{BMOA(X)}$$

for $n\geqslant n_0$ and $a\in\mathbb{D}$ satisfying $|\varphi(a)|\leqslant R$. Together with (7) this proves (4). □

It is now easy to complete the proof of Theorem 7.

Proof of Theorem 7. Let X and φ be as assumed. Then the operators V_n are weakly compact on $BMOA(X)$ for $n\geqslant 0$, by Lemma 8(3). Since the weakly compact operators form a closed operator ideal, it suffices to verify that

$$\|C_\varphi - C_\varphi V_n\|\to 0$$

as $n\to\infty$. Since by Smith's result φ satisfies conditions (1) and (3), this follows from Proposition 11. □

As a consequence, we obtain an analogue of Theorem 7 for $VMOA(X)$.

Corollary 12. *Let X be a reflexive Banach space and let φ be an analytic self-map of the unit disk such that $\varphi\in VMOA$. If C_φ is compact on VMOA, then C_φ is weakly compact on $VMOA(X)$.*

Proof. Let X and φ be as assumed. Then C_φ is compact on $BMOA$ by [29, Corollary 1.3], and C_φ is weakly compact on $BMOA(X)$ by Theorem 7. If (f_n) is a bounded sequence in $VMOA(X)$, then $(f_n\circ\varphi)$ has a weakly converging subsequence $(f_{n_k}\circ\varphi)$ in $BMOA(X)$.

By Corollary 5, the subsequence belongs to $VMOA(X)$, and hence it converges weakly to a function $g \in VMOA(X)$. Thus C_φ is weakly compact on $VMOA(X)$. $\square$

In the light of Fact 6 and Theorem 7 a complete characterization of the weakly compact composition operators on $BMOA(X)$ depends on whether all weakly compact composition operators on *BMOA* are compact or not. Unfortunately the answer to this question is not known for arbitrary composition operators C_φ (see, e.g., [12]). However, by combining with some partial positive results from the literature, we obtain the following consequence of Theorem 7.

Corollary 13. *Let φ be an analytic self-map of the unit disk such that φ satisfies one of the following conditions*:

(1) *φ is univalent, or*
(2) *$\varphi \in VMOA$ and $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle.*

Then C_φ is weakly compact on $BMOA(X)$ if and only if X is reflexive and C_φ is compact on BMOA.

Proof. Assume first that $\varphi : \mathbb{D} \to \mathbb{D}$ is univalent and C_φ is weakly compact on $BMOA(X)$. Then C_φ is weakly compact on *BMOA* and X is reflexive by Fact 6. It is well known that every bounded univalent map belongs to the Dirichlet space which in turn is included in *VMOA* (see, for instance, [18, p. 154]). Thus φ induces a weakly compact composition operator on *VMOA*. By [12, Theorem 1] and [29, Theorem 4.1], the operator C_φ is actually compact on *VMOA*. Since C_φ on *BMOA* is the second adjoint of C_φ on *VMOA* (cf. [12, p. 939]), we get that C_φ is compact also on *BMOA*.

The proof is similar in the case where $\varphi \in VMOA$ maps $\mathbb{D}$ inside a polygon inscribed in the unit circle. Here we apply a result by Tjani (see the proof of [32, Theorem 3.15], or [26, Corollary 5.4]) stating that if such a map induces a weakly compact composition operator on *VMOA*, then C_φ is compact on *VMOA*.

In both cases the converse statement follows from Theorem 7. $\square$

Remark 14. By modifying the proof of Theorem 7, one may obtain sufficient conditions for C_φ to belong to various operator ideals. As examples we briefly discuss weakly conditionally compact and completely continuous composition operators, which have been studied previously on various spaces of analytic functions (see [8,11,25]).

Recall that a linear map $T : X \to X$ is called weakly conditionally compact if for every sequence $(x_k) \subset X$ the sequence (Tx_k) admits a weakly Cauchy subsequence. Recall that T is completely continuous if it maps weakly Cauchy sequences to norm convergent sequences. Rosenthal's l^1-theorem [24, 2.e.5] implies that the identity operator of X is weakly conditionally compact if and only if X does not contain an isomorphic copy of l^1. A Banach space X is said to have the Schur property if its identity operator is completely continuous.

It is possible to modify the argument of Theorem 7 in the case where X does not contain a copy of l^1 or X has the Schur property, respectively. In fact, one may show that if C_φ

is compact on *BMOA*, then C_φ is weakly conditionally compact (respectively, completely continuous) on $BMOA(X)$, if X does not contain a copy of l^1 (respectively, X has the Schur property). A similar reasoning works for $VMOA(X)$. Recall that since the dual space H^1 of *VMOA* is separable, *VMOA* does not contain a copy of l^1. Thus every bounded sequence in *VMOA* admits a weakly Cauchy subsequence and every completely continuous linear operator on *VMOA* is compact. In particular, using Fact 6, we have that C_φ is completely continuous on $VMOA(X)$ if and only if C_φ is compact on *VMOA* and X has the Schur property. The details are left for the interested reader.

5. Weak vector-valued *BMOA*

In this section we discuss another interesting version of the vector-valued *BMOA*, the space $wBMOA(X)$ consisting of the weak X-valued *BMOA* functions. The purpose of this section is to demonstrate that $wBMOA(X)$ differs from the space $BMOA(X)$ considered earlier in this paper. Weak vector-valued *BMO* was earlier considered, e.g., in [4] and [22], and composition operators on various weak spaces were studied systematically in [8] by different methods.

Let $wBMOA(X)$ denote the space of analytic functions $f:\mathbb{D}\to X$ such that $x^*\circ f\in BMOA$ for every $x^*\in X^*$. The norm of $wBMOA(X)$ is given by

$$\|f\|_{wBMOA(X)}=\sup_{\|x^*\|\leqslant 1}\|x^*\circ f\|_{BMOA}.$$

Similarly, for $1\leqslant p<\infty$, let $wH^p(X)$ denote the space of analytic functions $f:\mathbb{D}\to X$ such that $x^*\circ f\in H^p$ for every $x^*\in X^*$, equipped with the norm

$$\|f\|_{wH^p(X)}=\sup_{\|x^*\|\leqslant 1}\|x^*\circ f\|_{H^p}.$$

Then $wBMOA(X)$ and $wH^p(X)$ are Banach spaces for every $1\leqslant p<\infty$ (cf. [8, Lemma 10]). Clearly

$$\|f\|_{wBMOA(X)}\leqslant\|f\|_{BMOA(X)}\quad\text{and}\quad\|f\|_{wH^p(X)}\leqslant\|f\|_{H^p(X)},$$

and the spaces coincide as sets whenever X is finite dimensional.

It is a general result due to Bonet, Domański and Lindström [8, Proposition 11] that the counterpart of Theorem 7 for $wBMOA(X)$ holds: If X is a reflexive Banach space and φ induces a compact composition operator on *BMOA*, then C_φ is weakly compact on $wBMOA(X)$. This raises the question whether $BMOA(X)$ is a closed subspace of $wBMOA(X)$ for (some) infinite dimensional X. Actually it turns out that this is never the case. In the case where X is a Hilbert space an example of this type was given in [22, Lemma 2.3] (see also [4]). We include here a concrete example based on a known multiplier result (due to Girela) and Dvoretzky's l_n^2-theorem, that applies to any infinite dimensional Banach space. We refer to, e.g., [15] for applications of Dvoretzky's theorem in parallel situations.

Example 15. For any infinite dimensional complex Banach space X there exists a sequence $(f_n)_{n=1}^\infty$ of analytic functions $f_n : \mathbb{D} \to X$ such that

$$\|f_n\|_{wBMOA(X)} \leqslant 1, \quad n \in \mathbb{N}, \quad \text{and} \quad \|f_n\|_{H^1(X)} \to \infty, \quad \text{as } n \to \infty.$$

In particular, the norms $\|\cdot\|_{wBMOA(X)}$ and $\|\cdot\|_{BMOA(X)}$, as well as the norms $\|\cdot\|_{wH^p(X)}$ and $\|\cdot\|_{H^p(X)}$, are not equivalent for any $1 \leqslant p < \infty$.

Proof. We construct the desired example using a known characterization of multipliers from l^2 to $BMOA$. A sequence $(a_k)_{k=0}^\infty$ is said to be a multiplier from l^2 to $BMOA$ if $\sum_{k=0}^\infty a_k b_k z^k \in BMOA$ for every $(b_k)_{k=0}^\infty \in l^2$. In that case we say that $(a_k)_{k=0}^\infty$ belongs to $(l^2, BMOA)$. By [18, Theorem 9.7], a sequence $(a_k)_{k=0}^\infty$ belongs to $(l^2, BMOA)$ if and only if

$$\sum_{k=0}^n k^2 |a_k|^2 = O(n^2),$$

as $n \to \infty$. Thus the sequence $(a_k)_{k=0}^\infty$ given by setting $a_0 = 0$ and $a_k = 1/\sqrt{k}$ for $k = 1, 2, \ldots$ belongs to $(l^2, BMOA)$. In particular, by the closed graph theorem there is a constant C such that

$$\left\| \sum_{k=1}^\infty \frac{b_k}{\sqrt{k}} z^k \right\|_{BMOA} \leqslant C \left(\sum_{k=1}^\infty |b_k|^2 \right)^{1/2}, \tag{8}$$

for $(b_k)_{k=1}^\infty \in l^2$. (I am indebted to S.V. Kislyakov for the comment that only Hardy's inequality [16, p. 48] and the duality argument from the proof of [18, Theorem 9.7] is needed to deduce that $(1/\sqrt{k})_{k=1}^\infty \in (l^2, BMOA)$.)

Let X be an infinite dimensional complex Banach space and $n \in \mathbb{N}$. By Dvoretzky's theorem [14, Theorem 19.1] there exists an n-dimensional subspace E_n of X and a linear isomorphism $J_n : l_n^2 \to E_n$ so that $\|J_n\| \leqslant 2$ and $\|J_n^{-1}\| = 1$. Let $x_k^{(n)} = J_n e_k^{(n)}$, where $e_k^{(n)}$ is the kth standard unit vector of l_n^2 for $k = 1, \ldots, n$. Define the analytic function $f_n : \mathbb{D} \to X$ by

$$f_n(z) = \sum_{k=1}^n \frac{x_k^{(n)}}{\sqrt{k}} z^k.$$

Then

$$\|f_n(re^{i\theta})\|_X \geqslant \left\| \sum_{k=1}^n \frac{e_k^{(n)}}{\sqrt{k}} (re^{i\theta})^k \right\|_{l_n^2} = \left(\sum_{k=1}^n \frac{r^{2k}}{k} \right)^{1/2}$$

for $0 < r < 1$, so that

$$\|f_n\|_{H^1(X)}^2 \geqslant \sup_{0<r<1} \left(\sum_{k=1}^n \frac{r^{2k}}{k} \right) = \sum_{k=1}^n \frac{1}{k} \geqslant \log n.$$

Suppose that $x^* \in X^*$ satisfies $\|x^*\|_{X^*} \leqslant 1$. Then $y_n^* = x^*|_{E_n} \in E_n^*$, and $J_n^* y_n^* \in (l_n^2)^*$ with $\|J_n^* y_n^*\|_{(l_n^2)^*} \leqslant \|J_n\| \|x^*\|_{X^*} \leqslant 2$, where J_n^* denotes the adjoint of J_n. We get from (8) that

$$\begin{aligned}\|x^* \circ f_n\|_{BMOA} &= \left\| \sum_{k=1}^{n} \frac{y_n^*(x_k^{(n)})}{\sqrt{k}} z^k \right\|_{BMOA} \\ &\leqslant C\left(\sum_{k=1}^{n} \left|y_n^*\left(x_k^{(n)}\right)\right|^2\right)^{1/2} = C\left(\sum_{k=1}^{n} \left|y_n^*\left(J_n e_k^{(n)}\right)\right|^2\right)^{1/2} \\ &= C\left(\sum_{k=1}^{n} \left|\left(J_n^* y_n^*\right)\left(e_k^{(n)}\right)\right|^2\right)^{1/2} = C\left\|J_n^* y_n^*\right\|_{(l_n^2)^*} \leqslant 2C.\end{aligned}$$

By taking the supremum over $x^* \in X$ satisfying $\|x^*\|_{X^*} \leqslant 1$, we get the estimate $\|f_n\|_{wBMOA(X)} \leqslant 2C$, where C is independent of n and X.

The fact that none of the norms are equivalent follows now from the continuous inclusions $BMOA(X) \subset H^1(X)$, $H^p(X) \subset H^1(X)$ and $wBMOA(X) \subset wH^p(X) \subset wH^1(X)$ that hold for every $1 \leqslant p < \infty$ by Hölder's inequality and the John-Nirenberg theorem (see [2] or [17]). □

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References

[1] J. Arazy, S.D. Fisher, J. Peetre, Möbius invariant function spaces, J. Reine Angew. Math. 363 (1985) 110–145.
[2] A. Baernstein II, Analytic functions of bounded mean oscillation, in: Aspects of Contemporary Complex Analysis, Proc. NATO Adv. Study Inst., Durham, 1979, Academic Press, London, 1980, pp. 3–36.
[3] O. Blasco, Boundary values of functions in vector-valued Hardy spaces and geometry on Banach spaces, J. Funct. Anal. 78 (1988) 346–364.
[4] O. Blasco, Operators from H^1 into a Banach space and vector valued measures, in: Geometric Aspects of Banach Spaces, in: London Math. Soc. Lecture Note Ser., vol. 140, Cambridge Univ. Press, Cambridge, UK, 1989, pp. 87–93.
[5] O. Blasco, A characterization of Hilbert spaces in terms of multipliers between spaces of vector-valued analytic functions, Michigan Math. J. 42 (1995) 537–543.
[6] O. Blasco, Vector-valued analytic functions of bounded mean oscillation and geometry of Banach spaces, Illinois J. Math. 41 (1997) 532–558.
[7] O. Blasco, Remarks on vector-valued BMOA and vector-valued multipliers, Positivity 4 (2000) 339–356.
[8] J. Bonet, P. Domański, M. Lindström, Weakly compact composition operators on analytic vector-valued function spaces, Ann. Acad. Sci. Fenn. Math. 26 (2001) 233–248.
[9] P.S. Bourdon, J.A. Cima, A.L. Matheson, Compact composition operators on BMOA, Trans. Amer. Math. Soc. 351 (1999) 2183–2196.
[10] A.V. Bukhvalov, A.A. Danilevich, Boundary properties of analytic and harmonic functions with values in a Banach space, Mat. Zametki 31 (1982) 203–214, 317. English translation: Math. Notes 31 (1982) 104–110.
[11] J.A. Cima, A. Matheson, Completely continuous composition operators, Trans. Amer. Math. Soc. 344 (1994) 849–856.
[12] J.A. Cima, A.L. Matheson, Weakly compact composition operators on VMO, Rocky Mountain J. Math. 32 (2002) 937–951.

[13] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.
[14] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge Univ. Press, Cambridge, UK, 1995.
[15] S.J. Dilworth, M. Girardi, Bochner vs. Pettis Norm: Examples and Results, in: Banach Spaces, Mérida, 1992, in: Contemp. Math., vol. 144, Amer. Math. Soc., Providence, RI, 1993, pp. 69–80.
[16] P.L. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.
[17] J.B. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.
[18] D. Girela, Analytic functions of bounded mean oscillation, in: Complex Function Spaces, Mekrijärvi, 1999, in: Univ. Joensuu Dept. Math. Rep. Ser., vol. 4, Univ. Joensuu, Joensuu, 2001, pp. 61–170.
[19] W. Hensgen, Hardy-Räume vektorwertiger Funktionen, Ph.D. thesis, Ludwig-Maximilians-Universität München, 1986.
[20] F. Holland, D. Walsh, Criteria for membership of Bloch space and its subspace, BMOA, Math. Ann. 273 (1986) 317–335.
[21] W.E. Hornor, J.E. Jamison, Isometrically equivalent composition operators on spaces of analytic vector-valued functions, Glasgow Math. J. 41 (1999) 441–451.
[22] B. Jacob, J.R. Partington, S. Pott, Admissible and weakly admissible observation operators for the right shift semigroup, Proc. Edinburgh Math. Soc. (2) 45 (2002) 353–362.
[23] Y. Katznelson, An Introduction to Harmonic Analysis, Dover, New York, 1976.
[24] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces. I, Springer-Verlag, Berlin, 1977.
[25] P. Liu, E. Saksman, H.-O. Tylli, Small composition operators on analytic vector-valued function spaces, Pacific J. Math. 184 (1998) 295–309.
[26] S. Makhmutov, M. Tjani, Composition operators on some Möbius invariant Banach spaces, Bull. Austral. Math. Soc. 62 (2000) 1–19.
[27] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[28] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[29] W. Smith, Compactness of composition operators on BMOA, Proc. Amer. Math. Soc. 127 (1999) 2715–2725.
[30] C.S. Stanton, Counting functions and majorization for Jensen measures, Pacific J. Math. 125 (1986) 459–468.
[31] K. Stephenson, Weak subordination and stable classes of meromorphic functions, Trans. Amer. Math. Soc. 262 (1980) 565–577.
[32] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, PhD thesis, Michigan State University, 1996.